Exchange-driven growth with a source and sink of particles

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Motivation

- Droplet growth via evaporation and recondensation
- Exchange of capital between economically interacting individuals
- Formation of smog
- Formation of planets from interstellar dust

Physical principles

- Exchange kernel is symmetric, $K(i,j) = K(j,i)$, and homogeneous, $K(ai, aj) = a^{2\lambda} K(i,j)$
- Spatial homogeneity
- Law of mass action

Modelling choices:

- Clusters of size 0
- Conservation of clusters
- Kernel homogeneity index: $K(i,j) = (ij)^{\lambda}$

Rate equation derivation

$$\frac{dc_k}{dt} =$$

$$K(k+1,j)c_{k+1}c_j - 2K(k,j)c_k c_j + K(j,k-1)c_{k-1}c_j$$

Rate equation derivation

$$\frac{dc_k}{dt} = \sum_{j=1}^{\infty} K(k+1,j)c_{k+1}c_j - 2K(k,j)c_k c_j + K(j,k-1)c_{k-1}c_j$$

Rate equation derivation

$$\frac{dc_k}{dt} = \sum_{j=1}^{\infty} K(k+1,j)c_{k+1}c_j - 2K(k,j)c_k c_j + K(j,k-1)c_{k-1}c_j$$
$$+ K(k+1,0)c_{k+1}c_0 - K(k,0)c_k c_0$$

ODE system

For clusters of size k = 0:

$$\frac{dc_0}{dt} = 0$$

and for $k = 1, ..., M-1$:

$$\frac{dc_k}{dt} = \sum_{j=1}^{M} K(k+1,j)c_{k+1}c_j - 2K(k,j)c_k c_j + K(j,k-1)c_{k-1}c_j$$

and for clusters of maximal size $k = M$:

$$\frac{dc_M}{dt} = \sum_{j=1}^{M-1} -2K(M,j)c_M c_j + K(j,M-1)c_{M-1}c_j$$
$$-2K(M,M)c_M^2 + K(M,M-1)c_{M-1}c_M$$

Scaling behaviour

Conservation of mass:

$$\int_0^\infty kc_k dk = 1$$

Typical size:

$$s(t) := \frac{M_2(t)}{M_1(t)}$$

where $M_\lambda(t) := \int_0^\infty k^\lambda c_k(t)dk$ is the λ-th moment.
Ben-Naim and Krapivsky, 2003, proposed a **self-similar** ansatz

$$c_k(t) = s(t)^\alpha F(z)$$

where $z = \frac{k}{s(t)}, \alpha = (3 - 2\lambda)^{-1}$ for $\lambda < \frac{3}{2}$, and showed that for $\lambda = 0$,

$$s(t) \sim t^{\frac{1}{3}}.$$

Scaling behaviour with source turned on

New rate equations:

$$\frac{dc_k}{dt} = \sum_j (...) + J\delta_{k,1}$$

so that

$$M_1 \sim t.$$

Krapivsky, 2015, showed

$$s(t) \sim t^2$$

for $K(i,j) = ij$. We generalised this to

$$s(t) \sim t^{\frac{2}{3-2\lambda}}$$

for $K(i,j) = (ij)^\lambda$.

Imposing a sink

Desorption:

$$\frac{dc_k}{dt} = \sum_j (...) + J\delta_{k,1} - \Gamma(k)c_k$$

Evaporation:

$$\frac{dc_k}{dt} = \sum_j (...) + J\delta_{k,1} + \begin{cases} \Gamma(k+1)c_{k+1} - \Gamma(k)c_k, & 0 \leq k \leq M-1 \\ -\Gamma(M)c_M, & k = M \end{cases}$$

where

$$\Gamma(k) = \gamma_0 k^\gamma.$$

Solver

$$\frac{d\mathbf{c}}{dt} = f(\mathbf{c})$$

where $\mathbf{c}(t) = \begin{pmatrix} c_0(t) \\ \vdots \\ c_M(t) \end{pmatrix}$ and

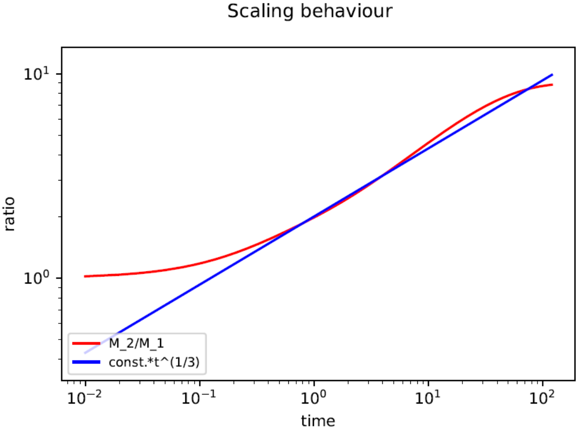
$$f(\mathbf{c}) = \begin{pmatrix} -\gamma_0 0^\gamma \\ \sum_j (K(2,j)c_2 c_j - ...) + J - \gamma_0 1^\gamma \\ \vdots \\ \sum_j (-2K(M,j)c_M c_j + ...) - \gamma_0 M^\gamma \end{pmatrix}$$

and a 'monodisperse' initial condition: $c_k(0) = \delta_{1,k}$.

- Predictor-corrector method with fixed timestep (3rd order local error in time).
- Python, within Jupyter notebooks.

Convergence and code validation

Scaling analysis:



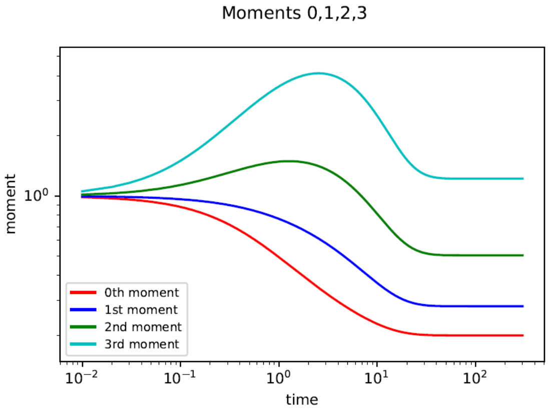
$$\text{Computation time} \sim \begin{cases} T & \text{simulated time} \\ M^2 & \text{numerical cutoff} \end{cases}$$

Experimentation: Scaling regimes and possibly phase transitions

- 'Waiting long enough' to observe scaling
- Don't confuse numerical regularisation with a physical process
- Is there a non-equilibrium steady state?

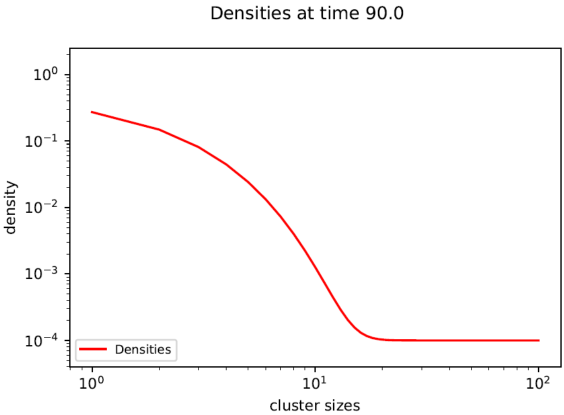
Conclusions

- Existence of steady states



- Settling time for the steady state increased with the ratio $\frac{J}{\gamma_0}$

- The steady state had an approximately exponential tail



- Larger systems may exhibit a phase transition (cf. Connaughton, Rajesh, Zaboronski, 2010).

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