# Ergodicity in Stochastic Process and Dynamical System

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LML, 2018

Definition: An observable is ergodic if its time average is the same as its expectation value.

$$\underbrace{\lim_{T\to\infty}\frac{1}{T}\int_0^T f(x(t))dt}_{\text{Time average of } f} = \underbrace{\int_\Omega f(x)P(x)dx}_{\text{Expectation value of } f}$$

- Ergodicity allows replacing time averages with expectation values.
- Studied in stochastic process and dynamical system.

# Outline

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# Random Variable

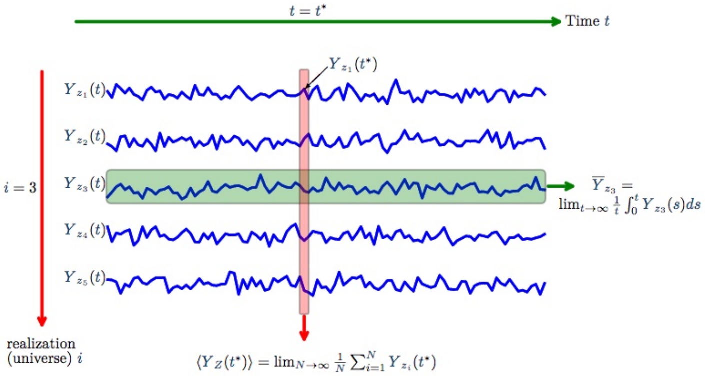
Random variable $Z$:

- set of possible values $\{z\}$
- probability distribution over $P(z)$

# Stochastic Process

A stochastic process $Y_z(t)$, is a family of random variables, one for each time, $t$.

# Stochastic process visualisation



# Example: Ornstein-Uhlenbeck process

$$dx = -\mu(x - x_0)dt + \sigma d\omega$$

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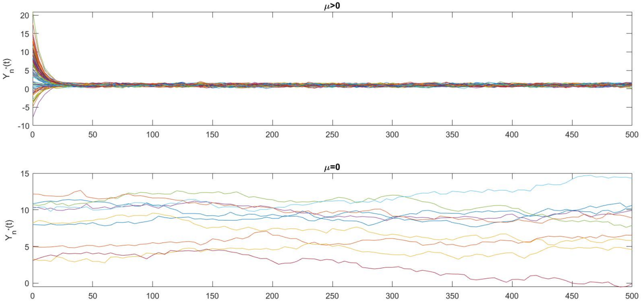
- Ergodic for $\mu > 0$
- Non-ergodic for $\mu \leq 0$

# Sample paths

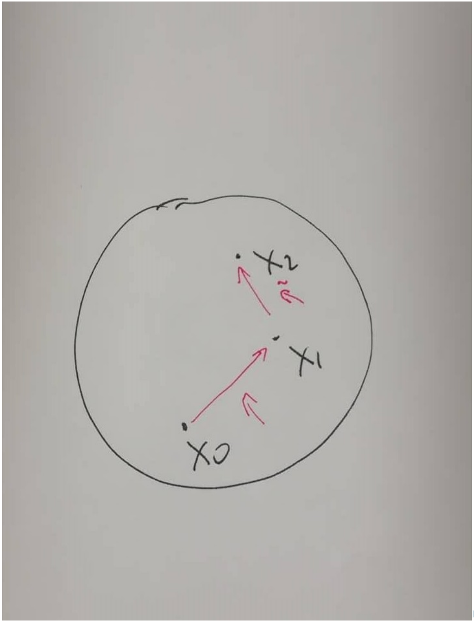


Ergodicity can be broken, for example if:

- the time evarage does not exist because the quantity $f$ grows
- or the expectation value diverges
- or the distribution $P(x(t))$ has no well-defined time limit

# Outline

# Dynamical system



# "Probability"

Probability, $P(x)$, in dynamical systems:

- Run the dynamical system for a long time with initial condition $x_0$
- The probability of an event $A$ is the relative frequency with which that event is observed.

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$x_{n+1} = (x_n + c) \mod 1$

- $c$ irrational: dynamics is ergodic
- $c$ rational: dynamics is non-ergodic

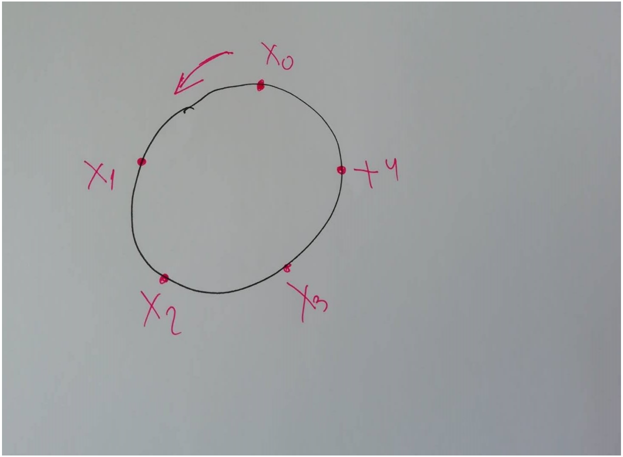
Rational $c$

$$x_{n+1} = \left(x_n + \frac{1}{5}\right) \mod 1$$

$$x_1 = \left(x_0 + \frac{1}{5}\right) \mod 1$$

$$x_2 = \left(x_1 + \frac{1}{5}\right) \mod 1$$

$\ldots$


X0
X1
X2
X3

Time average: $\frac{1}{5}\sum_{i=0}^{i=4} f(x_i)$

Expectation value : $\int_0^1 f(x)\underbrace{\frac{1}{5}\delta(x-x_i)}_{P(x)}dx$

# How ergodicity can be broken?

- time average does not exist
- sample space split into different components depending on initial condition $x_0$

# Outline

# Summary

Thoughts:

- Stochastic process: Birkhoff's equation holds for any trajectory.
- Dynamical system: Birkhoff's equation holds for any $x(t = 0)$.
- No time in random variable
- Dynamical systems has a closed space in mind (some blob we draw), whereas stochastic processes take place, typically, on the infinite real line, so you run into different problems.